An Elementary Proof of the Equivalence between 2D and 3D Classical Snakes and Geodesic Active Contours

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Abstract: Recently, Caselles et al. have shown in the equivalence between a classical snake problem of Kass et al. and a geodesic active contour model. The PDE derived from the geodesic problem gives an evolution equation for active contours which is very powerfull for image segmentation since changes of topology are allowed using the level set implementation. However in Caselles' paper the equivalence with classical snake is only shown for 2D images with 1D curves, by using concepts of Hamiltonian theory which have no meanings for active contours. This paper propose a proof using only elementary calculus of mathematical analysis. This proof is also valid in the 3D case for active surfaces.

Key-words: active contours, geodesics, PDE's

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Une preuve simple de l'équivalence entre les contours actifs classiques et les contours actifs géodésiques en 2D et 3D

Résumé : Les équations aux dérivées partielles (EDP) définissant l'évolution de courbe plane permettent, avec une implantation par ligne de niveau, un changement de topologie par rapport à la courbe initiale et sont de ce fait un outil puissant pour la segmentation d'objet dans une image. Récemment, Caselles et al. ont montrés l'équivalence entre les modèles de contours actifs classiques de Kass et al. et de contours actifs géodésiques qui définissent une EDP particulière d'évolution de courbe. Cependant la preuve proposée par Caselles n'est valable que pour la segmentation d'objet d'une image 2D avec des courbes 1D et fait appel à des concepts de la théorie hamiltonienne, sans aucun sens physique pour les contours actifs. Ce papier propose une preuve utilisant uniquement des calculs élémentaires d'analyse mathématique, valables aussi pour la segmentation d'image 3D à l'aide de surface.

Mots-clés : contours actifs, géodésiques, EDP

1 Introduction

Boundary detection using active contour models has been extensively studied during the last decade. The classical segmentation method introduced by Kass et al [5] for boundary detection consists of estimating snakes by minimizing an energy. Recently, new active contour models based on curve evolution have been introduced by Caselles et al. [1] and by Malladi et al [6]. Assuming that the n-dimensional curve (n=1) or surface (n=2) is represented as the zero level set of a (n+1)-dimensional function, the model is then given by a geometric flow driven by the so-called mean curvature motion. Automatic changes in the topology are then allowed, while they are not handled in the classical snake approach. This property generates a novel interest from researchers for active contour models and shape recovery.
In [2], Caselles et al have shown that the classical snake problem (with no elasticity constraint) is equivalent to finding a geodesic curve in a Riemannian space, where the metric depends on the image. This equivalence is only shown for 2D images with 1D curves by using the Maupertuis' principle of least action from dynamical systems. Then the Fermat's principle is applied to fix a free parameter.
The goal of this paper is to show that using concepts of the Hamiltonian theory is not necessary to show the equivalence between snakes and geodesic active contours. We show the equivalence using only elementary calculus of mathematical analysis. Moreover, this proof is also valid in the 3D case for active surfaces. In the work of Caselles in [3], the geometric 3D approach is extended from the 2D one proposed in [2], but the surface evolution model is not connected with the classical snake energy one.
This paper is organized as follows. In section 2 we recall the classical energy model (without the elasticity constraint) and the problem of geodesic computation in a Riemannian space, according to the metric derived from the image. In section 3 we present the proof of the equivalence of these two problems in the 2D case. In section 4 the extension of the proof to the 3D case is derived. Concluding remarks are given in section 5.

2 Two active contour models

Denote $C(q) : [0,1] \to \mathrm{R}^2$ a piecewise $\mathcal{C}^1$ parametrized curve and $I : \Omega \to \mathrm{R}^+$ an image in which we want to detect object boundaries, Ω is an open set of R^2.

2.1 The snake model

We consider the following snake energy:

$$J_1(C) = \int_0^1 |C'(q)|^2 dq + \lambda \int_0^1 g(|\nabla I(C(q))|)^2 dq \tag{1}$$

where g is a function which defines an edge detector. Therefore, g is a monotonic decreasing regular function such that $\lim_{t \to +\infty} g(t) = 0$. Solving the problem of snakes amounts to

finding the curve C that minimizes J_1. The first term in right-hand side of (1) controls the smoothness of the curve. We only consider the rigidity term (with first order derivative), dropping the elasticity term (with second order derivative) which is commonly added in the general case [5]. In [2], the authors justify the fact that they are rather redundant in the final equivalent geodesic model developed.
The second term is the external energy and attracts the contour towards the edges of the object in the image I.
An initial contour is deformed towards the boundary of the object to be detected. The main drawback of this approach is that this energy model does not handle changes in the topology of the evolving contour when direct implementation is performed. Moreover this energy depends on the parametrization of the curve resulting in a non intrinsic approach.

2.2 The geodesic model

In [2], Caselles et al. have shown using deep hamiltonian concepts that minimizing the criterion J_1 of (1) is equivalent to minimize the criterion J_2 defined by:

$$J_2(C) = 2\int_0^1 |C'(q)|.g(|\nabla I(C(q))|)dq \tag{2}$$

J_2 defines a problem of geodesic computation in a Riemannian space, according to a metric induced by the image I. Minimizing J_2 to detect an object consists in finding the path of minimal new length which takes into account image characteristics. The curve evolution is obtained by computing the corresponding steepest-descent flow of J_2. This geometric flow is driven by the mean curvature motion. This model allows automatic changes in the topology when implemented by using the level-sets based numerical algorithm [7]. This is not possible with the classical snake approach which consists in minimizing J_1.

2.3 Goal of the paper

The goal of the paper is to show by using elementary calculus of mathematical analysis that solving the problem of snakes defined in (1) is equivalent to a geodesic computation in a Riemannian space defined in (2), without Maupertuis and Fermat's principles of the Hamiltonian theory.

More precisely, we define two problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ by:

$$(\mathcal{P}_1) \qquad \inf_{C\in\mathcal{C}} \int_0^1 |C'(q)|^2 dq + \int_0^1 g(|\nabla I(C(q))|)^2 dq \tag{3}$$

$$(\mathcal{P}_2) \qquad \inf_{C\in\mathcal{C}} 2\int_0^1 |C'(q)|.g(|\nabla I(C(q))|)dq \tag{4}$$

where $\mathcal{C} = \{C : [0,1] \to \mathrm{R}^2, C$ is piecewise $\mathrm{C}^1\}$. Problem $(\mathcal{P}_1)$ corresponds to the criterion J_1 with $\lambda = 1$, which will be assumed for sake of simplicity, without loss of generality.

The goal of this work is to show that $(\mathcal{P}_1) = (\mathcal{P}_2)$. From the inequality $a^2+b^2 \geq 2ab\ \forall a, b$ we have

$$\int_0^1 |C'(q)|^2 dq + \int_0^1 g(|\nabla I(C(q))|)^2 dq \geq 2\int_0^1 |C'(q)|.g(|\nabla I(C(q))|)dq$$

from which we deduce $(\mathcal{P}_1) \geq (\mathcal{P}_2)$. The purpose of the next section is to show the reverse inequality $(\mathcal{P}_1) \leq (\mathcal{P}_2)$ which is a difficult part for proving that $(\mathcal{P}_1) = (\mathcal{P}_2)$.

3 Proof of the inequality $(\mathcal{P}_1) \leq (\mathcal{P}_2)$

The main idea of the proof is the following: let $C \in \mathcal{C}$ be a given curve, we show that there exists an other curve $C_{**} \in \mathcal{C}$ such that:

$$2\int_0^1 |C'(q)|.g(|\nabla I(C(q))|)dq \geq \int_0^1 |C'_{**}(q)|^2 dq + \int_0^1 g(|\nabla I(C_{**}(q))|)^2 dq \tag{5}$$

Taking the infimum over $C \in \mathcal{C}$ successively on the right-hand side and the left-hand side of the above inequality, $(\mathcal{P}_1) \leq (\mathcal{P}_2)$ is derived.

3.1 Notations, hypotheses and definitions

In this section we define notations and hypotheses.

Denote Γ the set of the image edges. We assume that $\Gamma = \bigcup_{j \in J} C_j, C_j \in \mathcal{C}$, J is finite or countable, C_j is piecewise C^1.

For the intensity image, we assume the following hypothesis:
$I : \Omega \in \mathrm{R}^2 \to \mathrm{R}$ such that

$$(x,y) \to |\nabla I(x,y)| \in \mathrm{C}^0(\Omega - \Gamma) \tag{6}$$

$$\lim_{d((x,y),\Gamma)\to 0} |\nabla I(x,y)| = +\infty \tag{7}$$

where d is the distance function.

In image processing, an edge is defined as points (x, y) such that $|\nabla I(x,y)| = +\infty$. In this article, we need a more precise definition. We assume that a curve $C_0 \in \Gamma$ is an edge of the image I if and only if there exists a non negative ε_0 such that

$$\forall \varepsilon < \varepsilon_0, \exists \alpha_\varepsilon, R_\varepsilon, R_\varepsilon \geq \alpha_\varepsilon > 0, \lim_{\varepsilon\to 0} \alpha_\varepsilon = \lim_{\varepsilon \to 0} R_\varepsilon = 0$$

with

$$
\begin{array}{lll}
|\nabla I(x,y)| & \geq \frac{1}{\varepsilon}, & \text{if } d((x,y),C_0) \leq \alpha_\varepsilon \qquad (8) \\
|\nabla I(x,y)| & \leq \varepsilon, & \text{if } d((x,y),C_0) \geq R_\varepsilon \qquad (9) \\
|\nabla I(x,y)| & \leq |\nabla I(x',y')|, & \text{if } \left\{ \begin{array}{l} d((x,y),C_0) \geq R_\varepsilon \\ d((x',y'),C_0) \leq R_\varepsilon \end{array} \right. \qquad (10)
\end{array}
$$

Denote

$$
\begin{array}{lll}
V_\varepsilon^1 & = & \{(x,y) \in \mathrm{R}^2 / d((x,y),C_0) \leq \alpha_\varepsilon\} \\
V_\varepsilon^2 & = & \{(x,y) \in \mathrm{R}^2 / \alpha_\varepsilon \leq d((x,y),C_0) \leq R_\varepsilon\} \\
V_\varepsilon^3 & = & \{(x,y) \in \mathrm{R}^2 / d((x,y),C_0) \geq R_\varepsilon\}
\end{array}
$$

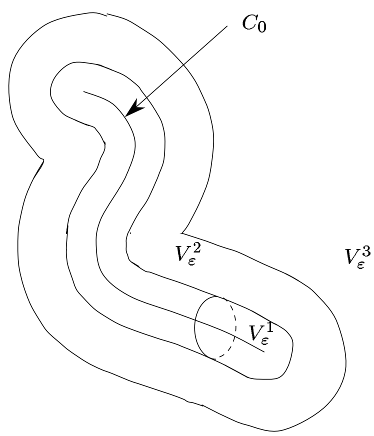

Figure 1 : example of a curve C_0 with associated sets V_ε^i

In order to specify edges by zero values rather than by infinite values, which is easier to handle, we define a function $g : \mathrm{R}^+ \rightarrow \mathrm{R}_*^+$ such that:

$$g(0) = 1 \qquad (11)$$

$$\lim_{t \rightarrow +\infty} g(t) = 0 \qquad (12)$$

$$g(.) \text{ is regular monotonic decreasing} \qquad (13)$$

$$(x,y) \rightarrow g(|\nabla I(x,y)|) \text{ is Lipschitz with Lipschitz constant } K \qquad (14)$$

A typical example of such function g is given by $g(t) = \frac{1}{1+t^2}$. We equally note:

$$W(C) = g(|\nabla I(C)|) \qquad (15)$$

$$W(x,y) = g(|\nabla I(x,y)|) \qquad (16)$$

3.2 Proof of $(\mathcal{P}_1) \leq (\mathcal{P}_2)$

The proof relies on the construction of a curve C_{**} satisfying (5). Let $C \in \mathcal{C}$ be a given curve. Without loss of generality, we can assume that with $|C'(t)| \neq 0 \quad a.e.$ We first parametrize C by its arc length $s = \tau(q) = \int_0^q |C'(t)|dt$. Define

$$C_*(s) = C(\tau^{-1}(s)) \tag{17}$$

we have

$$|C'_*(s)| = 1 \ a.e. \tag{18}$$

and (2) is written as follows:

$$J_2(C) = 2\int_0^L W(C_*(s))ds \tag{19}$$

where $L = \int_0^1 |C'(t)|dt$=length of C. The next step consists in reparametrize the curve $C_*(s)$ and to construct a curve $C_{**}(t)$ satisfying $W(C_{**}(s)) = |C'_{**}(s)| \ a.e.$ Denote h the solution of the differential equation:

$$\begin{cases} h'(t) = W(C_*(h(t)) \\ h(0) = 0 \end{cases} \tag{20}$$

According to the hypothesis (14) , (20) has a unique solution $h \in \mathrm{C}^1$ with $h' > 0$. Define

$$\begin{aligned} s &= h(t) \\ C_{**}(t) &= C_*(h(t)) \end{aligned} \tag{21}$$

then we have

$$ds = h'(t)dt = W(C_*(h(t)))dt = W(C_{**}(t))dt \tag{22}$$

$$|C'_{**}(t)| = h'(t)|C'_*(h(t))| = h'(t) = W(C_{**}(t)) \tag{23}$$

and

$$J_2(C) = 2\int_0^{h^{-1}(L)} W^2(C_{**}(t))dt \tag{24}$$

Denoting $T = h^{-1}(L)$, we obtain using (23):

$$J_2(C) = 2\int_0^T W^2(C_{**}(t))dt = \int_0^T \left[W^2(C_{**}(t)) + |C'_{**}(t)|^2\right] dt \tag{25}$$

We have shown, starting from $C \in \mathcal{C}$ that there exists a curve $C_{**} \in \mathcal{C}$ such that

$$2\int_0^1 W(C(t)|C'(t)|dt = \int_0^T \left[W^2(C_{**}(t)) + |C'_{**}(t)|^2\right] dt \tag{26}$$

In order to conclude that $(\mathcal{P}_1) \leq (\mathcal{P}_2)$ we have to examine the two cases $T \geq 1$ and $T < 1$.

3.2.1 $T \geq 1$

In this case, since the integrands are non negative, we have:

$$\begin{aligned} J_2(C) = \int_0^T \left[W^2(C_{**}(t)) + |C'_{**}(t)|^2\right] dt \geq \\ \int_0^1 \left[W^2(C_{**}(t)) + |C'_{**}(t)|^2\right] dt \geq (\mathcal{P}_1) \end{aligned} \tag{27}$$

hence

$$2\int_0^1 W(C(q))|C'(q)|dq \geq (\mathcal{P}_1) \tag{28}$$

Taking the infimum over $C \in \mathcal{C}$, we deduce $(\mathcal{P}_1) \leq (\mathcal{P}_2)$.

3.2.2 $T < 1$

This case is not as easy as the previous one. The aim is to extend the curve $C_{**}(t)$ on the interval $(T, 1)$ without any additional contribution for J_2. To achieve this, we use the definition of an edge and the hypotheses on g stated in paragraph 3.1. We assume for clarity that I contains only one contour. According to (8-10), denote ε such that $\varepsilon < \varepsilon_0$. We examine the two cases whether the intersection $C_{**} \cap V_\varepsilon^1$ is empty or not. Recall that $J_2 = \int_0^T \left[W^2(C_{**}(t)) + |C'_{**}(t)|^2\right] dt$. In each case, we derive a lower bound for J_2 as:

$$J_2(C) \geq (\mathcal{P}_1) + f(\varepsilon) \text{ with } \lim_{\varepsilon \to +\infty} f(\varepsilon) = 0 \tag{29}$$

- First case: $\exists b \in C_{**} \cap V_\varepsilon^1$
  Without loss of generality, we assume that $b = C_{**}(T)$. We extend C_{**} on $(0, T]$ by defining C_{**}^ε such that:

  $$C_{**}^\varepsilon(t) = \begin{cases} C_{**}(t) \text{ if } t \in [0, T] \\ b = C_{**}(T) \text{ if } t \in (T, 1] \end{cases}$$

  Since $(C_{**}^\varepsilon)' = 0$ on $(T, 1]$, it is easy to get

  $$J_2(C) = \int_0^1 \left[W^2(C_{**}^\varepsilon(t)) + |\,(C_{**}^\varepsilon)'\,(t)|^2\right] dt - (1-T)W^2(b) \tag{30}$$

But $b \in V_\varepsilon^1$, then we have according to (7) and (13):

$$W^2(b) = g^2(|\nabla I(b)|) \leq g^2(\frac{1}{\varepsilon})$$

Then

$$J_2(C) \geq \int_0^1 \left[W^2(C_{**}^\varepsilon(t)) + |\,(C_{**}^\varepsilon)'\,(t)|^2\right] dt - (1-T)g^2(\frac{1}{\varepsilon}) \tag{31}$$

Hence

$$J_2(C) \geq (\mathcal{P}_1) - (1-T)g^2(\frac{1}{\varepsilon}) \tag{32}$$

- Second case: $C_{**} \cap V_\varepsilon^1 = \emptyset$
  Denote $b = C_{**}(T)$ as in the previous case, and b_ε defined as

$$d(b, b_\varepsilon) = d(b, V_\varepsilon^1)$$

and

$$C_\varepsilon(t) = C_{**}(t) + \overrightarrow{bb_\varepsilon}.$$

C_{**} is defined on $[0, T]$. We extend C_ε on the interval $(T, 1]$ using

$$\tilde{C}_\varepsilon(t) = \begin{cases} C_\varepsilon(t) \text{ if } t \in [0, T] \\ b_\varepsilon \text{ if } t \in (T, 1] \end{cases}$$

Hence, by the definition of $\tilde{C}_\varepsilon$ and noting that $C'_\varepsilon = C'_{**}$ we have

$$\int_0^1 \left[W^2(\tilde{C}_\varepsilon(t)) + |\tilde{C}'_\varepsilon(t)|^2\right] dt = \tag{33}$$

$$\int_0^T \left[W^2(C_\varepsilon(t)) + |C'_{**}(t)|^2\right] dt + (1-T)W^2(b_\varepsilon)$$

and we deduce in a similar way as in the previous case that

$$\int_0^T \left[W^2(C_\varepsilon(t)) + |C'_{**}(t)|^2\right] dt \geq \tag{34}$$

$$\int_0^1 \left[W^2(\tilde{C}_\varepsilon(t)) + |(\tilde{C}'_\varepsilon(t)|^2\right] dt - (1-T)g^2(\frac{1}{\varepsilon})$$

The difference between the left-hand side of (34) and $J_2(C)$ is that we have C_ε rather than C_{**} in the first term. The following technical calculus are derived in order to estimate the difference $\int_0^T \left[W^2(C_\varepsilon(t)) - W^2(C_{**}(t)\right] dt$ and so to obtain an inequality similar to (31) in this case. We define two intervals on $[0,T]$ denoted Δ_2 and Δ_3:

$$\begin{aligned}\Delta_2 &= \{t \in [0,T]/C_{**}(t) \in V_\varepsilon^2\}\\ \Delta_3 &= \{t \in [0,T]/C_{**}(t) \in V_\varepsilon^3\}\end{aligned}$$

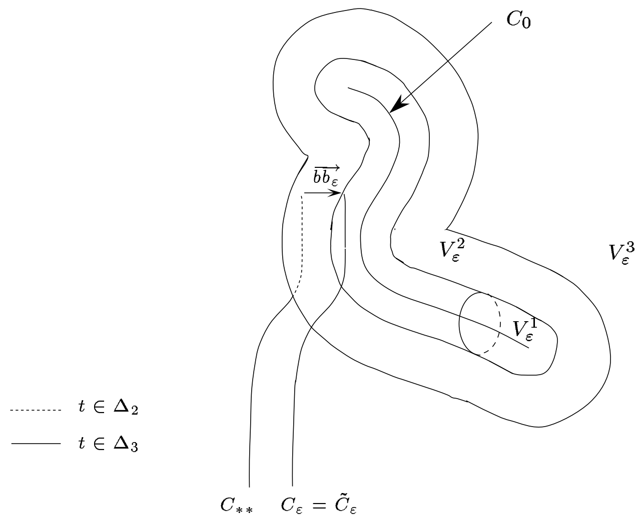

Figure 2: example of curve C_0, C_{**} and C_ε

Since Δ_2 and Δ_3 form a partition of [0,T] we have:

$$\int_0^T W^2(C_\varepsilon(t)) = \int_{\Delta_2} W^2(C_\varepsilon(t)) + \int_{\Delta_3} W^2(C_\varepsilon(t)) \tag{35}$$

In order to calculate the integrals of the right-hand side of (35), each one must be again decomposed over three sets:

$$\begin{aligned}\Delta_2^i &= \{t \in \Delta_2/C_\epsilon(t) \in V_\varepsilon^i\}\\ \Delta_3^i &= \{t \in \Delta_3/C_\epsilon(t) \in V_\varepsilon^i\}, \qquad i = 1,2,3\end{aligned} \tag{36}$$

– Calculus over Δ_2

$$\int_{\Delta_2} W^2(C_\varepsilon(t))dt = \int_{\Delta_2^1} W^2(C_\varepsilon(t))dt + \int_{\Delta_2^2} W^2(C_\varepsilon(t))dt + \int_{\Delta_2^3} W^2(C_\varepsilon(t))dt \tag{37}$$

We now estimate each integral over each subset Δ_2^i:

1. Over Δ_2^1, we have

$$\int_{\Delta_2^1} W^2(C_\varepsilon(t))dt = \int_{\Delta_2^1} W^2(C_{**}(t))dt + \int_{\Delta_2^1} \left[W^2(C_\varepsilon(t))dt - W^2(C_{**}(t))dt\right] \tag{38}$$

Due to (14),

$$|W^2(C_\varepsilon(t)) - W^2(C_{**}(t))| \leq K|C_\varepsilon(t) - C_{**}(t)| \tag{39}$$

from which we deduce, recalling that $C_{**}(t) \in V_\varepsilon^2$ and $C_\varepsilon(t) \in V_\varepsilon^1$:

$$\int_{\Delta_2^1} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_2^1} W^2(C_{**}(t))dt + KT(\alpha_\varepsilon + R_\varepsilon) \tag{40}$$

2. In a similar way we obtain:

$$\int_{\Delta_2^2} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_2^2} W^2(C_{**}(t))dt + 2KTR_\varepsilon \tag{41}$$

3. The integral over Δ_2^3 is more complex because $C_{**}(t) \in V_\varepsilon^2$ and $C_\varepsilon(t) \in V_\varepsilon^3$. Adding a new point

$$\hat{C}(t) = \partial V_\varepsilon^2 \cap [C_{**}(t), C_\varepsilon(t)]\,.$$

We can write

$$\int_{\Delta_2^3} W^2(C_\varepsilon(t))dt = \int_{\Delta_2^3} W^2(C_{**}(t))dt + \int_{\Delta_2^3} \left[W^2(\hat{C}(t))dt - W^2(C_{**}(t))dt\right] + \int_{\Delta_2^3} \left[W^2(C_\varepsilon(t))dt - W^2(\hat{C}(t))dt\right] \tag{42}$$

We have

$$\int_{\Delta_2^3} |W^2(\hat{C}(t)) - W^2(C_{**}(t))|dt \leq KT|\hat{C} - C_{**}| \leq KTR_\varepsilon \tag{43}$$

$$\int_{\Delta_2^3} \left[W^2(C_\varepsilon(t)) - W^2(\hat{C}(t))\right] dt = \int_{\Delta_2^3} \left[g^2(|\nabla I(C_\varepsilon(t))|) - g^2(|\nabla I(\hat{C(t)})|)dt\right] \tag{44}$$

According to (9) and (13) and by Taylor formula there exists a constant K' such that:

$$\int_{\Delta_2^3} \left[W^2(C_\varepsilon(t)) - W^2(\hat{C}(t)) \right] dt \leq 2.K'.T.\varepsilon \tag{45}$$

Then

$$\int_{\Delta_2^3} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_2^3} W^2(C_{**}(t))dt + 2KTR_\varepsilon + 2K'T\varepsilon \tag{46}$$

Finally over Δ_2 we have shown that

$$\int_{\Delta_2} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_2} W^2(C_{**}(t))dt + A_\varepsilon^2 \tag{47}$$

where $A_\varepsilon^2 = 2KTR_\varepsilon + 2K'T\varepsilon \underset{\varepsilon \to 0}{\to} 0$

– Calculus over Δ_3
We proceed as for Δ_2

$$\int_{\Delta_3} W^2(C_\varepsilon(t))dt = \int_{\Delta_3^1} W^2(C_\varepsilon(t))dt + \int_{\Delta_3^2} W^2(C_\varepsilon(t))dt + \int_{\Delta_3^3} W^2(C_\varepsilon(t))dt \tag{48}$$

1. Over Δ_3^1, we have

$$\int_{\Delta_3^1} W^2(C_\varepsilon(t))dt = \int_{\Delta_3^1} W^2(C_{**}(t))dt + \int_{\Delta_3^1} \left[W^2(C_\varepsilon(t)) - W^2(C_{**}(t)) \right] dt \tag{49}$$

$C_{**}(t) \in V_\varepsilon^3$ and $C_\varepsilon(t) \in V_\varepsilon^1$ so with hypothesis (10)

$$|\nabla I(C_{**}(t))| \leq |\nabla I(C_\varepsilon(t))| \quad \forall t \in \Delta_3^1$$

Since g is monotonic decreasing we have

$$\int_{\Delta_3^1} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_3^1} W^2(C_{**}(t))dt \tag{50}$$

2. For the same reasons we get

$$\int_{\Delta_3^2} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_3^2} W^2(C_{**}(t))dt \tag{51}$$

3. Over Δ_3^3, we have

$$\int_{\Delta_3^3} W^2(C_\varepsilon(t))dt = \int_{\Delta_3^3} W^2(C_{**}(t))dt + \int_{\Delta_3^3} \left[W^2(C_\varepsilon(t)) - W^2(C_{**}(t))\right] dt \tag{52}$$

$C_{**}(t)$ and $C_\varepsilon(t)$ being in V_ε^3 we have

$$\int_{\Delta_3^3} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_3^3} W^2(C_{**}(t))dt + 2K'T\varepsilon \tag{53}$$

Finally on Δ_3 we have shown that

$$\int_{\Delta_3} W^2(C_\varepsilon(t))dt \leq \int_{\Delta_3} W^2(C_{**}(t))dt + 2K'T\varepsilon \tag{54}$$

Therefore from (35), (47) and (54) we get

$$\int_0^T W^2(C_\varepsilon(t))dt \leq \int_0^T W^2(C_{**}(t))dt + A_\varepsilon^2 + 2K'T\varepsilon \tag{55}$$

and from (34) and (55):

$$\int_0^T \left[W^2(C_{**}(t)) + |C'_{**}(t)|^2\right] dt \geq \int_0^1 \left[W^2(\tilde{C}_\varepsilon(t)) + |\tilde{C}'_\varepsilon(t)|^2\right] dt - (1-T)g(\frac{1}{\varepsilon}) - A_\varepsilon^2 - 2K'T\varepsilon \tag{56}$$

Letting $A_\varepsilon = (1-T)g(\frac{1}{\varepsilon}) + A_\varepsilon^2 + 2K'T\varepsilon$, (56) can be written as

$$J_2(C) \geq (\mathcal{P}_1) - A_\varepsilon. \tag{57}$$

By coupling the results (32) and (57) of the first and the second case (according to the intersection $C_{**} \cap V_\varepsilon^1$), we obtain :

$$J_2(C) \geq (\mathcal{P}_1) - \sup\left((1-T)g^2(\frac{1}{\varepsilon}), A_\varepsilon\right) \tag{58}$$

that is

$$2\int_0^1 |C'(q)|.g(|\nabla I(C(q))|)dq \geq (\mathcal{P}_1) - \sup\left((1-T)g^2(\frac{1}{\varepsilon}), A_\varepsilon\right) \tag{59}$$

and we conclude that $(\mathcal{P}_1) \leq (\mathcal{P}_2)$ by doing $\varepsilon \to 0$ and by taking the infimum over $C \in \mathcal{C}$ in (59).

That achieves the proof in the case $T < 1$.

4 Extension to surfaces

The proof in the 3D case for surfaces is similar to the one previously derived in the 2D case for curves. We only sketch below the main points of the proof in this case.
Denote $\mathcal{D}$ a regular set in R^2, $S(u,v)$ for (u,v) in $\mathcal{D}$ be a parametrized surface. We denote

$$E = \left|\frac{\partial S}{\partial u}\right|^2 \tag{60}$$

$$F = \frac{\partial S}{\partial u}\frac{\partial S}{\partial v} \tag{61}$$

$$G = \left|\frac{\partial S}{\partial v}\right|^2 \tag{62}$$

the coefficients of the first fundamental form of $S(u,v)$. The surface element is $dA = \sqrt{EG - F^2}dudv$, J_2 is

$$J_2(S) = 2\int_{\mathcal{D}} W\left(S(u,v)\right)\sqrt{EG-F^2}dudv \tag{63}$$

and

$$J_1(S) = 2\int_{\mathcal{D}} \left[W^2\left(S(u,v)\right) + \left(EG - F^2\right)\right] dudv \tag{64}$$

We assume that for all (u,v) in $\mathcal{D}$, $\frac{\partial S}{\partial u}$ and $\frac{\partial S}{\partial v}$ are non colinear vectors, that is $EG - F^2 > 0$, $\forall u, v$.
$(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are now stated as follows:

$$(\mathcal{P}_1): \inf_{S\in\mathcal{S}} J_1(S), \qquad (\mathcal{P}_2): \inf_{S\in\mathcal{S}} J_2(S) \tag{65}$$

where $\mathcal{S} = \{S : \mathcal{D} \to \mathrm{R}^3, S \text{ piecewise } \mathrm{C}^1\}$. As for the previous case $(\mathcal{P}_1) \geq (\mathcal{P}_2)$. In order to prove that $(\mathcal{P}_1) \leq (\mathcal{P}_2)$, we proceed as in paragraph 3. Given any surface S, we show using reparametrizations and extensions, the existence of a surface $\overline{S}$ such that

$$J_2(S) \geq J_2(\overline{S}) \tag{66}$$

from which we deduce that $(\mathcal{P}_2) \geq (\mathcal{P}_1)$. The proof of (66) is similar to the 2D case's one, the only specific point lies on the reparametrization defined in (20).
A change of variable is defined by a one to one function

$$\begin{aligned} H : \mathrm{R}^2 &\rightarrow \mathrm{R}^2 \\ (s,t) &\rightarrow (h_1(s,t), h_2(s,t)) = (u,v) \end{aligned} \tag{67}$$

J_2 becomes

$$J_2(S) = 2 \int_{H^{-1}(\mathcal{D})} W\left(S(h_1(s,t), h_2(s,t))\right) \sqrt{EG - F^2}\, |det \nabla H|\, dsdt \tag{68}$$

In order to obtain a surface S_{**} such that $S_{**}(s,t) = S\left(h_1(s,t), h_2(s,t)\right)$ and $W\left(S_{**}(s,t)\right) = \sqrt{E_{**}G_{**} - F_{**}^2}$ where E_{**}, G_{**}, F_{**} are defined by (60-62) for $S = S_{**}$, let consider the PDE:

$$det\,(\nabla H) = \frac{W\left(S(h_1,h_2)\right)}{\sqrt{EG - F^2(h_1,h_2)}} \tag{69}$$

It can be shown [4] that adding the right boundary conditions, there exists a diffeomorphism solution of (69). For this function H, J_2 is written as follows:

$$J_2(S) = \int_{H^{-1}(\mathcal{D})} \left[W^2\left(S_{**}(s,t)\right) + \left(E_{**}G_{**} - F_{**}^2\right)(s,t)\right] dsdt \tag{70}$$

We conclude by examining the difference between the measures of $\mathcal{D}$ and $H^{-1}(\mathcal{D})$ and by extending S_{**} if necessary (cf § 3.2).

5 Concluding remarks

- The proof is developed in the case of a single contour in the image. Of course the demonstration is also available in the case of a countable number of contours.
- We have only shown that J_1 and J_2 have the same infimal values but the question of the existence and the unicity of the infimum remains open. If there exist, we can only show that a solution of $(\mathcal{P}_1)$ is also a solution of $(\mathcal{P}_2)$, but nothing can be said for the reverse.

References

[1] V. Caselles, F.Catte, T. Coll, and F.Dibos. A geometric model for active contours. *Numerische Mathematik*, 66:1–31, 1993.

[2] V. Caselles, R. Kimmel, and G. Sapiro. On geodesic active contours. *Int. Journal of Computer Vision*, 22(1):61–79, February 1997.

[3] V. Caselles, R. Kimmel, G. Sapiro, and C. Sbert. Minimal surfaces: A geometric three dimensional segmentation approach. *Numerishe Math.*, 1998. to appear.

[4] B. Dacorogna and J. Moser. On a partial differential equation involving the Jacobian determinant. *Annales de l'Institut Henri Poincaré, Analyse non Linéaire*, 7:1–26, 1990.

[5] M. Kass, A. Witkin, and D.Terzopoulos. Snakes : Active contour models. *International Journal of Computer Vision*, 1:321–331, 1988.

[6] R. Malladi, J.A. Sethian, and B.C. Vermuri. Shape modeling with front propagation: A level set approach. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 17(2), Feb 1995.

[7] S. Osher and J.A. Sethian. Front propagating with curvature dependant speed: Algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

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